

GLOBAL SMOOTH SOLUTIONS FOR A HYPERBOLIC CHEMOTAXIS MODEL ON A NETWORK

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ABSTRACT. In this paper we study a semilinear hyperbolic-parabolic system modeling biological phenomena evolving on a network composed by oriented arcs. We prove the existence of global (in time) smooth solutions to this problem. The result is obtained by using energy estimates with suitable transmission conditions at nodes.

1. Introduction

In this paper we consider a semilinear hyperbolic-parabolic system which describes chemosensitive movements of bacteria, cells or other microorganisms on an artificial scaffold.

A mathematical approach to phenomena involving chemotaxis has been largely developed in the last thirty years, see [17, 21, 19], mostly by means of the study of the Patlak-Keller-Segel system [11]; this model was constituted by a parabolic equation governing the evolution of the density of cells, and a parabolic or elliptic one for the evolution of concentration of chemoattractant.

On the contrary, here, following [8, 9], we consider a model where the evolution of the density of cells is described by a hyperbolic system, coupled with a parabolic equation for the chemoattractant, which in one space dimension reads

$$(1.1) \quad \begin{cases} u_t + \lambda v_x = 0, \\ v_t + \lambda u_x = \phi_x u - \beta v, \\ \phi_t = D\phi_{xx} + au - b\phi. \end{cases} \quad x \in I, t \geq 0,$$

The unknown u stands for the concentration of cells, v is their average flux and ϕ is the concentration of chemoattractant produced by the cells themselves; the source term $\phi_x u$ is the nonlinear chemotactic term. As regard to the various parameters, $D > 0$ is the diffusion coefficient of chemoattractant, $a \geq 0$ and $b > 0$ are respectively its production and degradation rates, $\beta > 0$ is the friction coefficient of substrate; finally, each individual can move at a constant velocity, whose modulus is $\lambda \geq 0$, towards right or left along the axis.

Hyperbolic models have been recently introduced [5, 6, 14, 21] since they yield a more realistic finite speed of propagation, in contrast with the parabolic ones, and allow a better observation of the phenomena during the transitory. Models like (1.1) were proposed in [22, 8], introducing the chemotactic term in the Cattaneo equation, and later their solutions were studied in [15, 16, 9].

1991 *Mathematics Subject Classification.* Primary 35R02; Secondary 35Q92, 35L50, 35M33.

Key words and phrases. nonlinear hyperbolic systems, networks, transmission conditions, global existence of solutions, chemotaxis.

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Here we consider the one dimensional model (1.1) cast on a network formed by n nodes N_ν and m oriented arcs connecting the nodes, I_i ; moreover, each arc is characterized by a typical velocity λ_i and a diffusion coefficient D_i . On each arc I_i we consider the triple of unknowns (u_i, v_i, ϕ_i) .

This model arises as a preliminary tool in tissue-engineering research, to describe the process of dermal wound healing: the fibroblasts are seeded on a polymeric scaffold and they move along it to fill the wound, driven by chemotaxis [13, 18, 23]. In our mathematical model the arcs mimic the fibers of the scaffold and they intersect at nodes and u_i, ϕ_i are the densities of fibroblasts and chemoattractant on each of them.

The aim of this paper is to give a proof of global existence of smooth solutions to problem (1.1) complemented with initial, boundary and transmission conditions at nodes. The rigorous statement of the problem is presented in Section 2.

Since the Cauchy and the Neumann problems were considered in [9], the crucial point of our study is in the transmission conditions, which heavily characterize the problem, since they are the coupling among the solutions on different arcs. So far, our analysis is based on the formulation of suitable transmission conditions at nodes, also in the preliminary proof of local existence. More precisely, we impose transmission conditions which guarantee the conservation of the fluxes, both for the hyperbolic system and the parabolic equation; in this way the energy of the linearized homogeneous version of the system decays in time. Section 3 is devoted to the motivation and the derivation of these conditions.

We mention that hyperbolic models on networks have been previously studied in [7, 4, 24]; moreover a parabolic chemotaxis model on networks was studied in [1], with continuity conditions at node. Finally, in [2] the authors treat the same our model from a numerical point of view, with slightly different transmission conditions.

In Section 4 we prove the first result of this paper, namely the existence and uniqueness of local solutions. This result is obtained by means of the linear contraction semigroups theory coupled with the abstract theory of nonhomogeneous and semilinear evolution problems; in fact the right position of transmission conditions allows to use the properties of m -dissipative linear operators.

Finally, in Section 5, we prove the existence of global solutions in the case of small (in a suitable norm) initial data. The proof of this result needs some further conditions at node providing the tools for controlling the growth of the unknowns' traces.

2. Statement of the problem

We consider a connected graph $G = (\mathcal{N}, \mathcal{A})$ composed by a set \mathcal{N} of n nodes, $\mathcal{N} = \{N_\nu : \nu \in \mathcal{P} = \{1, 2, \dots, n\}\}$ and a set \mathcal{A} of m oriented arcs, $\mathcal{A} = \{I_i : i \in \mathcal{M} = \{1, 2, \dots, m\}\}$. Moreover we denote by a_j , $j \in \mathcal{J} = \{1, 2, \dots, l\}$ the external points of the graph.

Since we are on an oriented network, for each node N_ν we consider the set of incoming arcs $\mathcal{A}_{in}^\nu = \{I_i : i \in \mathcal{I}^\nu\}$ and the set of the outgoing ones $\mathcal{A}_{out}^\nu = \{I_i : i \in \mathcal{O}^\nu\}$; let $\mathcal{M}^\nu = \mathcal{I}^\nu \cup \mathcal{O}^\nu$.

Each oriented arc I_i is a compact one dimensional interval; if I_i is an external arc, connecting a boundary point a_j to a node N_ν , then it has the form $I_i = [a_j, N_\nu]$ when $i \in \mathcal{I}^\nu$ and the form $I_i = [N_\nu, a_j]$ when $i \in \mathcal{O}^\nu$; if I_i is an internal arc, connecting the nodes N_ν and N_μ then it has form $I_i = [N_\nu, N_\mu]$ if $i \in \mathcal{O}^\nu \cap \mathcal{I}^\mu$.

A function f defined on \mathcal{A} is a m-pla of functions f_i , $i \in \mathcal{M}$, each one defined on I_i ; $L^2(\mathcal{A}) = \cup_{i \in \mathcal{M}} L^2(I_i)$, $H^s(\mathcal{A}) = \cup_{i \in \mathcal{M}} H^s(I_i)$ and

$$\|f\|_2 := \sum_{i \in \mathcal{M}} \|f_i\|_2, \quad \|f\|_{H^s} := \sum_{i \in \mathcal{M}} \|f_i\|_{H^s}.$$

We consider the evolution on the graph G of the following one dimensional problem

$$(2.1) \quad \begin{cases} \partial_t u_i + \lambda_i \partial_x v_i = 0, \\ \partial_t v_i + \lambda_i \partial_x u_i = u_i \partial_x \phi_i - \beta_i v_i, \\ \partial_t \phi_i = D_i \partial_{xx} \phi_i + a u_i - b \phi_i, \end{cases} \quad x \in I_i, t \geq 0, i \in \mathcal{M},$$

where $\lambda_i, a \geq 0$, $b, D_i, \beta_i > 0$. Actually, the coefficients a and b could depend on the arc I_i in consideration, but $\frac{a_i}{b_i}$ should be constant for $i \in \mathcal{M}$.

We complement the system with the initial conditions

$$(2.2) \quad u_{i0}, v_{i0} \in H^1(I_i), \quad \phi_{i0} \in H^2(I_i) \quad \text{for } i \in \mathcal{M}.$$

As regard to the boundary conditions, at each outer point a_i we set the null flux conditions

$$(2.3) \quad v_i(a_i, t) = 0, \quad t > 0,$$

$$(2.4) \quad \phi_{ix}(a_i, t) = 0 \quad t > 0.$$

Besides, at each node N_ν we impose the following transmission conditions for $\phi_i(N_\nu, t)$

$$(2.5) \quad \begin{cases} D_i \phi_{ix}(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu, t) - \phi_i(N_\nu, t)), \quad i \in \mathcal{I}^\nu, \quad t > 0, \\ -D_i \phi_{ix}(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu, t) - \phi_i(N_\nu, t)), \quad i \in \mathcal{O}^\nu, \quad t > 0, \\ \alpha_{ij}^\nu \geq 0, \quad \alpha_{ij}^\nu = \alpha_{ji}^\nu \quad \text{for all } i, j \in \mathcal{M}^\nu, \end{cases}$$

which imply the continuity of the flux at node, for all $t > 0$,

$$\sum_{i \in \mathcal{I}^\nu} D_i \phi_{ix}(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} D_i \phi_{ix}(N_\nu, t).$$

In similar way we impose some transmission conditions for the unknowns $v_i(N_\nu, t)$ and $u_i(N_\nu, t)$

$$(2.6) \quad \begin{cases} -\lambda_i v_i(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu (u_j(N_\nu, t) - u_i(N_\nu, t)), \quad i \in \mathcal{I}^\nu, \quad t > 0, \\ \lambda_i v_i(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu (u_j(N_\nu, t) - u_i(N_\nu, t)), \quad i \in \mathcal{O}^\nu, \quad t > 0, \\ K_{ij}^\nu \geq 0, \quad K_{ij}^\nu = K_{ji}^\nu \quad \text{for all } i, j \in \mathcal{M}^\nu. \end{cases}$$

The above conditions ensure the conservation of the flux of the density of cells at each node N_ν , for $t > 0$,

$$\sum_{i \in \mathcal{I}^\nu} \lambda_i v_i(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} \lambda_i v_i(N_\nu, t).$$

We notice that the previous equality corresponds to the conservation of the total mass

$$\sum_{i \in \mathcal{M}} \int_{I_i} u_i(x, t) \, dx = \sum_{i \in \mathcal{M}} \int_{I_i} u_{0i}(x) \, dx ,$$

which means that no death nor birth of individuals occurs during the observation.

The constraints on the coefficients in the transmission conditions will be widely motivated in the next section.

Finally, we impose the following compatibility conditions

$$(2.7) \quad u_{i0}, v_{i0}, \phi_{i0} \text{ satisfy conditions (2.3)-(2.6) for all } i \in \mathcal{M} .$$

Provided the above conditions, we will prove the existence and uniqueness of a local solution to problem (2.1)-(2.7),

$$(u, v) \in C([0, T]; H^1(\mathcal{A})) \cap C^1([0, T], L^2(\mathcal{A})) , \quad \phi \in C([0, T]; H^2(\mathcal{A})) \cap C^1([0, T], L^2(\mathcal{A})) .$$

The final aim of the paper is the study of existence of global solutions under the assumption of smallness of the data. This question involves the proof of estimates for the traces of the unknowns u_i at nodes N_ν , which can be derived under the further assumption that, for all $\nu \in \mathcal{P}$, for some $k \in \mathcal{M}^\nu$, the coefficients K_{ik}^ν are non null for all $i \in \mathcal{M}^\nu$, $i \neq k$.

3. Transmission conditions

In the present section we explain the derivation of the transmission conditions (2.5) and (2.6) at nodes. Such conditions have to guarantee two properties of the model; first, the conservation of the flux of the density of cells, at each node N_ν

$$(3.1) \quad \sum_{i \in \mathcal{I}^\nu} \lambda_i v_i(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} \lambda_i v_i(N_\nu, t)$$

and the conservation of the flux of the parabolic equation

$$(3.2) \quad \sum_{i \in \mathcal{I}^\nu} D_i \phi_{ix}(N_\nu, t) = \sum_{i \in \mathcal{O}^\nu} D_i \phi_{ix}(N_\nu, t) ,$$

for $t > 0$. Moreover, it is necessary to deal with dissipative conditions; in other words, the sum of the m energies of the linear version of the hyperbolic systems,

$$(3.3) \quad \begin{cases} \partial_t u_i + \lambda_i \partial_x v_i = 0 \\ \partial_t v_i + \lambda_i \partial_x u_i = -\beta_i v_i \end{cases}$$

and the sum of the ones of the homogeneous parabolic equations

$$(3.4) \quad \partial_t \phi_i = D_i \partial_{xx} \phi_i - b \phi_i$$

have to decay in time. It is clear, by using the integration by parts, that the dissipation of such energies

$$E_1(t) = \sum_{i \in \mathcal{M}} \int_{I_i} (u_i^2(x, t) + v_i^2(x, t)) \, dx , \quad E_2(t) = \sum_{i \in \mathcal{M}} \int_{I_i} \phi_i^2(x, t) \, dx ,$$

is strictly linked with the following sign properties of the terms at nodes

$$(3.5) \quad \Gamma_1^\nu(t) = \sum_{i \in \mathcal{I}^\nu} \lambda_i v_i u_i(N_\nu, t) - \sum_{i \in \mathcal{O}^\nu} \lambda_i v_i u_i(N_\nu, t) \geq 0 , \quad \nu \in \mathcal{P} ,$$

$$(3.6) \quad \Gamma_2^\nu(t) = \sum_{i \in \mathcal{I}^\nu} D_i \phi_i \phi_{ix}(N_\nu, t) - \sum_{i \in \mathcal{O}^\nu} D_i \phi_i \phi_{ix}(N_\nu, t) \geq 0 , \quad \nu \in \mathcal{P} .$$

In particular, the above conditions ensure that the linear unbounded operators appearing in the problems (3.3), (3.4) are dissipative; this property is crucial to

apply the theory of linear contraction semigroups, to prove the existence of local solutions, in the next section.

In order to derive transmission conditions which imply (3.5), (3.6), for some $\nu \in \mathcal{P}$, we first consider the simple case of two arcs, I_1 incoming in N_ν and I_2 outgoing from N_ν . Here the flux conservation condition (3.1), together with inequality (3.5), reads

$$\lambda_1 v_1(N_\nu, t)(u_1 - u_2)(N_\nu, t) = \lambda_2 v_2(N_\nu, t)(u_1 - u_2)(N_\nu, t) \geq 0 ;$$

a possible condition to make this inequality true, is to link the values $v_i(N_\nu, t)$ and $u_i(N_\nu, t)$ as follows:

$$\lambda_1 v_1(N_\nu, t) = \lambda_2 v_2(N_\nu, t) = K^\nu (u_1(N_\nu, t) - u_2(N_\nu, t)) , \quad \text{for some } K^\nu \geq 0 .$$

Arguing in similar way, using (3.2) and (3.6), we obtain the transmission conditions for ϕ at node

$$D_1 \phi_{x1}(N_\nu, t) = D_2 \phi_{x2}(N_\nu, t) = \alpha^\nu (\phi_2 - \phi_1)(N_\nu, t) , \quad \text{for some } \alpha^\nu \geq 0 .$$

Let us also notice that such kind of conditions for parabolic equations, were introduced in [10] in the description of passive transport through biological membranes and they are known as Kedem- Katchalsky permeability conditions.

In the case of m arcs intersecting in N_ν , the continuity of the flux (3.1) and the inequality (3.5) provide the following conditions at node, for $j \neq i$

$$-\sum_{i \in \mathcal{I}^\nu} \lambda_i v_i(N_\nu, t)(u_j - u_i)(N_\nu, t) + \sum_{i \in \mathcal{O}^\nu} \lambda_i v_i(N_\nu, t)(u_j - u_i)(N_\nu, t) \geq 0 \quad \text{for all } j \in \mathcal{M}^\nu ;$$

hence some relations among the values $v_i(N_\nu, t)$ and the jumps $(u_j - u_i)(N_\nu, t)$, for $i, j \in \mathcal{M}^\nu$, are expected to be asked.

We assume that the terms $v_i(N_\nu, t)$ are linear combinations of the jumps $u_i(N_\nu, t) - u_j(N_\nu, t)$, $j \in \mathcal{M}^\nu$:

$$-\lambda_i v_i(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu (u_j(N_\nu, t) - u_i(N_\nu, t)) , \quad i \in \mathcal{I}^\nu , \quad (3.7)$$

$$\lambda_i v_i(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} K_{ij}^\nu (u_j(N_\nu, t) - u_i(N_\nu, t)) , \quad i \in \mathcal{O}^\nu .$$

Inserting the above positions in the flux continuity relation we obtain

$$\sum_{i, j \in \mathcal{M}^\nu} K_{ij}^\nu (u_i - u_j) (N_\nu, t) = 0 ;$$

so, we obtain a first constrain to the coefficients

$$\sum_{i \in \mathcal{M}^\nu} (K_{ij}^\nu - K_{ji}^\nu) = 0 \quad \text{for all } j \in \mathcal{M}^\nu .$$

Now we consider the dissipation condition (3.5) which reads

$$\sum_{i, j \in \mathcal{M}^\nu} K_{ij}^\nu (u_i - u_j) (N_\nu, t) u_i(N_\nu, t) \geq 0 ;$$

sufficient conditions to guarantee the above inequality are

$$K_{ij}^\nu = K_{ji}^\nu, K_{ij}^\nu \geq 0 \quad \text{for all } i, j \in \mathcal{M}^\nu .$$

The corresponding conditions on the coefficients α_{ij}^ν follow by similar calculations.

Finally, we notice that in [2] the authors study our problem by a numerical point of view, introducing transmission conditions for the Riemann invariants of the hyperbolic part of the system, $w_i^\pm = \frac{1}{2}(u_i \pm v_i)$, which, in some cases, are equivalent to the present ones.

4. Local existence

Let $Y = \cup_{i \in \mathcal{M}} (L^2(I_i))^2$ $X = L^2(\mathcal{A})$.

We consider the linear operator $A_1 : D(A_1) \rightarrow Y$,

$$(4.1) \quad \begin{aligned} D(A_1) &= \{U = (u, v) \in \cup_{i \in \mathcal{M}} (H^1(I_i))^2 : (2.3), (2.6) \text{ hold} \} \\ A_1 U &= \{(-\lambda_i v_{ix}, -\lambda_i u_{ix})\}_{i \in \mathcal{M}} , \end{aligned}$$

and the linear operator $A_2 : D(A_2) \rightarrow X$,

$$(4.2) \quad \begin{aligned} D(A_2) &= \{\phi \in H^2(\mathcal{A}) : (2.4), (2.5) \text{ hold} \} \\ A_2(\phi) &= \{D_i \phi_{ixx} - b\phi_i\}_{i \in \mathcal{M}} . \end{aligned}$$

We will obtain the existence of local solutions to problem (2.1)-(2.7) by the fixed point technique, combining the local solutions of the two disjointed problems

$$(4.3) \quad \begin{cases} U \in C([0, T]; D(A_1)) \cap C^1([0, T]; Y) \\ U'(t) = A_1 U(t) + F(t, U(t)) , \quad t \in [0, T] , \\ U(0) = (u_0, v_0) \in D(A_1) , \end{cases}$$

where $F(t, U(t)) = \{(0, f_i(t) u_i(t) - \beta_i v_i(t))\}_{i \in \mathcal{M}}$ and f is a suitable given function to be specified below, and

$$(4.4) \quad \begin{cases} \phi \in C([0, T]; D(A_2)) \cap C^1([0, T]; X) \\ \phi'(t) = A_2 \phi(t) + g(t) , \quad t \in [0, T] , \\ \phi(0) = \phi_0 \in D(A_2) , \end{cases}$$

where g is a suitable given function to be specified below.

We have some results concerning the solutions of such problems. In order to simplify the notations, we will give the proofs in the case of a graph composed by a single node N and m arcs I_i connecting that node to the external points a_i , $i \in \mathcal{M} = \{1, 2, \dots, m\}$. In the general case there are no major differences when treating integrals on the external arcs; on the other hand, two transmission terms arise when integrating on internal arcs $I_i = (N_\nu, N_\mu)$, each one corresponding to a node. Then, the sum of all the transmission terms at each node of the graph can be treated separately, as in the case of a single node.

Proposition 4.1. *Let $T < 1$, let $g \in C([0, T]; H^1(\mathcal{A})) \cap C^1([0, T], L^2(\mathcal{A}))$, $M > \sup_{[0, T]} \|g(t)\|_{H^1}$ and $K > \|\phi_0\|_{H^2} + 4M$; then there exists a unique solution to problem (4.4) and*

$$\sup_{t \in [0, T]} \|\phi(t)\|_{H^2} \leq K .$$

Moreover, $\phi \in H^1((0, T); H^1(\mathcal{A}))$.

Proof. We are going to prove that A_2 generates a semigroup in X .

First, A_2 is dissipative in X :

$$\begin{aligned}
 (A_2\phi, \phi) &= \sum_{i \in \mathcal{M}} \int_{I_i} (D_i\phi_i \partial_{xx}\phi_i - b\phi_i^2) \, dx = - \sum_{i \in \mathcal{M}} \int_{I_i} (D_i\phi_{ix}^2 + b\phi_i^2) \, dx \\
 (4.5) \quad &+ \sum_{i \in \mathcal{I}} D_i\phi_{ix}(N, t)\phi_i(N, t) - \sum_{i \in \mathcal{O}} D_i\phi_{ix}(N, t)\phi_i(N, t) \\
 &= -\frac{1}{2} \sum_{i, j \in M} \alpha_{ij}(\phi_j(N, t) - \phi_i(N, t))^2 - \int_{I_i} (D_i\phi_{ix}^2 + b\phi_i^2) \, dx,
 \end{aligned}$$

where we used the transmission conditions (2.5).

Moreover, for all $\varphi \in L^2(\mathcal{A})$ there exists $\phi \in D(A_2)$ such that $\phi - A_2\phi = \varphi$, i.e. A_2 is a m-dissipative operator in X . In order to prove this fact we introduce the bilinear form $a(\phi, \psi) : (H^1(\mathcal{A}))^2 \rightarrow \mathbb{R}$

$$\begin{aligned}
 (4.6) \quad a(\phi, \psi) &= \sum_{i \in \mathcal{M}} \int_{I_i} (D_i\phi_{ix}\psi_{ix} + (1+b)\phi_i\psi_i) \, dx \\
 &- \sum_{i, j \in M} \alpha_{ij}(\phi_j(N) - \phi_i(N))\psi_i(N);
 \end{aligned}$$

it is easy to verify that the form is continuous and coercive. Then, by the Lax-Milgram theorem, we know that, for all $\varphi \in L^2(\mathcal{A})$, there exists a unique $\phi \in H^1(\mathcal{A})$ such that, for all $\psi \in H^1(\mathcal{A})$ it holds

$$a(\phi, \psi) = \sum_{i \in \mathcal{M}} \int_{I_i} \varphi_i \psi_i \, dx;$$

taking $\psi_i \in H_0^1(I_i)$ for all $i \in \mathcal{M}$, we obtain that $\phi_{ix} \in H^1(I_i)$, then

$$\begin{aligned}
 (4.7) \quad &\sum_{i \in \mathcal{M}} \int_{I_i} (-D_i\phi_{ixx} + (1+b)\phi_i)\psi_i \, dx + \sum_{i \in \mathcal{I}} D_i(\phi_{ix}(N)\psi_i(N) - \phi_{ix}(a_i)\psi_i(a_i)) \\
 &- \sum_{i \in \mathcal{O}} D_i(\phi_{ix}(N)\psi_i(N) - \phi_{ix}(a_i)\psi_i(a_i)) - \sum_{i, j \in M} \alpha_{ij}(\phi_j - \phi_i)(N)\psi_i(N) \\
 &= \sum_{i \in \mathcal{M}} \int_{I_i} \varphi_i \psi_i \, dx.
 \end{aligned}$$

The above equality holds for all $\psi_i \in C_0^\infty(I_i)$, then

$$-\phi_{ixx} + (1+b)\phi_i = \varphi_i \quad \text{a.e. for all } i \in \mathcal{M}$$

and moreover, thanks to suitable choices of $\psi_i(N), \psi_i(a_i)$, we obtain that ϕ satisfies the right boundary conditions to belong to $D(A_2)$.

Then A_2 is m-dissipative [3] and generates a contraction semigroup $\mathcal{T}_2(t)$ in X ; since $g \in C^1([0, T], L^2(\mathcal{A}))$ we can apply the theory for nonhomogeneous problems in [3] to conclude that there exists a unique solution to the problem (4.4), given by

$$\phi(t) = \mathcal{T}_2(t)\phi_0 + \int_0^t \mathcal{T}_2(t-s)g(s) \, ds.$$

We set

$$\mathcal{F}(t) := \int_0^t \mathcal{T}_2(t-s)g(s) \, ds.$$

$\mathcal{F} \in C^1([0, T]; L^2(\mathcal{A}))$ and

$$(4.8) \quad \mathcal{F}'(t) = \int_0^t \mathcal{T}_2(s) g'(t-s) ds + \mathcal{T}_2(t) g(0) ;$$

moreover $\mathcal{F} \in C([0, T]; D(A_2))$ and $A_2 \mathcal{F}(t) = \mathcal{F}'(t) - g(t)$, see [3].

Hence

$$(4.9) \quad \begin{aligned} \|\phi(t)\|_{D(A_2)} &\leq \|\phi_0\|_{D(A_2)} + \|\mathcal{F}(t)\|_X + \|A_2 \mathcal{F}(t)\|_X \\ &\leq \|\phi_0\|_{D(A_2)} + \int_0^t \|g(s)\|_X ds + \|\mathcal{F}'(t)\|_X + \|g(t)\|_X. \end{aligned}$$

Now, using (4.8) we have

$$(4.10) \quad \begin{aligned} \|\phi(t)\|_{D(A_2)} &\leq \|\phi_0\|_{D(A_2)} + \|g(0)\|_X + \|g(t)\|_X \\ &\quad + t \left(\sup_{s \in [0, t]} \|g'(s)\|_X + \sup_{s \in [0, t]} \|g(s)\|_X \right) \end{aligned}$$

and, thanks to the condition on T , we have

$$\sup_{t \in [0, T]} \|\phi(t)\|_{H^2} \leq K ,$$

where K is the quantity introduced in the statement of the theorem.

To prove the last claim, it is sufficient to prove that there exists $C > 0$ such that, for all $0 < t_1 < t_2 < T$,

$$(4.11) \quad \int_{t_1}^{t_2} \|\phi_x(t+h) - \phi_x(t)\|_2^2 dt \leq C|h|^2 ,$$

for all $h \in \mathbb{R}$ with $|h| < \min\{t_1, T - t_2\}$.

Let $\Delta^h \psi(t) := \psi(t+h) - \psi(t)$; using the equation we can write

$$\int_{t_1}^{t_2} \int_{I_i} ((\Delta^h \phi_i)_t \Delta^h \phi_i - D_i(\Delta^h \phi_i)_{xx} \Delta^h \phi_i + \Delta^h g_i \Delta^h \phi_i - (\Delta^h \phi_i)^2) dx dt = 0 ;$$

then we have

$$(4.12) \quad \begin{aligned} &\sum_{i \in \mathcal{M}} \left(\int_{I_i} (\Delta^h \phi_i(t_2))^2 dx + \int_{t_1}^{t_2} \int_{I_i} (\Delta^h \phi_{i_x})^2 dx dt \right) \\ &\leq C \int_{t_1}^{t_2} \left(\sum_{i \in \mathcal{I}} D_i(\Delta^h \phi_{i_x})(\Delta^h \phi_i)(N, t) - \sum_{i \in \mathcal{O}} D_i(\Delta^h \phi_{i_x})(\Delta^h \phi_i)(N, t) \right) dt \\ &\quad + C \sum_{i \in \mathcal{M}} \left(\int_{I_i} (\Delta^h \phi_i(t_1))^2 dx + \int_{t_1}^{t_2} \int_{I_i} (\Delta^h g_i)^2 dx dt \right) , \end{aligned}$$

hence inequality (4.11) follows thanks to nonpositivity of the first term on the right hand side (as in (4.5)), since $\phi, g \in C^1((0, T); L^2(\mathcal{A}))$.

□

In order to treat problem (4.3) and to prove the results of the next proposition, we need the following lemma.

Lemma 4.1. *Let $W = (w, z) \in \cup_{i \in \mathcal{M}} (C_0^\infty(I_i))^2$; there exists a unique $U = (u, v) \in D(A_1)$ such that $(I - A_1)U = W$.*

Proof. Let $\theta_i = 1$ if $i \in \mathcal{I}$ and $\theta_i = -1$ if $i \in \mathcal{O}$. We consider the elliptic problem

$$(4.13) \quad \begin{cases} -\lambda_i^2 u_{ixx} + u_i = -\lambda_i z_{ix} + w_i \\ u_{ix}(a_i) = 0, \quad \theta_i \lambda_i^2 u_{ix}(N) = \sum_{j \in \mathcal{M}} K_{ij} (u_j(N) - u_i(N)) ; \end{cases}$$

in the proof of the previous proposition, in the steps to obtain the m -dissipativity of A_2 , we showed that such problem has a unique solution u whose components u_i , in the present case, belong to $C^\infty(I_i)$. Now we set $v_i = -z_i - \lambda_i u_{ix}$; then $v_i \in C^\infty(I_i)$, $\lambda_i v_{ix} + u_i = w_i$ and $v_i(a_i) = 0$, $-\theta_i \lambda_i v_i(N) = \sum_{j \in \mathcal{M}} K_{ij} (u_j(N) - u_i(N))$. \square

Notice that, if $f \in C([0, T]; H^1(\mathcal{A}))$, then $F(t, U(t)) = f(t)u(t) - \beta v(t)$ is a globally Lipschitz function in $E := \cup_{i \in \mathcal{M}} (H^1(I_i))^2$, with Lipschitz constant $L_F = L_F \left(\sup_{t \in [0, T]} \|f(t)\|_{H^1} \right)$; more precisely

$$\sup_{[0, T]} \|F(t, U_1(t)) - F(t, U_2(t))\|_E \leq L_F \sup_{[0, T]} \|U_1(t) - U_2(t)\|_E .$$

Proposition 4.2. *Let $f \in C([0, T_1]; H^1(\mathcal{A})) \cap H^1((0, T_1); L^2(\mathcal{A}))$, $K > \sup_{[0, T_1]} \|f(t)\|_{H^1}$,*

$M > 2(\|u_0\|_{H^1} + \|v_0\|_{H^1})$ and $T < \min\{T_1, (2L_F(K))^{-1}\}$; then there exists a unique solution to problem (4.3) and

$$\sup_{t \in [0, T]} \|U(t)\|_E \leq M .$$

Proof. First we prove that A_1 generates a contraction semigroup in Y . A_1 is a dissipative operator in Y : let $U \in D(A_1)$

$$(4.14) \quad \begin{aligned} (A_1 U, U) &= \sum_{i \in \mathcal{M}} \int_{I_i} (-\lambda_i v_{ix} u_i - \lambda_i u_{ix} v_i) \\ &= - \left[\sum_{i \in \mathcal{I}} \lambda_i v_i(N) u_i(N) - \sum_{i \in \mathcal{O}} \lambda_i v_i(N) u_i(N) \right] ; \end{aligned}$$

now, using the transmission conditions (2.6) we have

$$(4.15) \quad (A_1 U, U) = -\frac{1}{2} \sum_{i, j \in \mathcal{M}} K_{ij} (u_j(N) - u_i(N))^2 .$$

In order to prove that A_1 is a m -dissipative operator in Y , we introduce the bilinear form $a : D(A_1) \times D(A_1) \rightarrow \mathbb{R}$

$$a(U, \bar{U}) = \sum_{i \in \mathcal{M}} \int_{I_i} ((\lambda_i v_{ix} + u_i)(\lambda_i \bar{v}_{ix} + \bar{u}_i) + (\lambda_i u_{ix} + v_i)(\lambda_i \bar{u}_{ix} + \bar{v}_i)) dx ;$$

a is continuous and, using the boundary and transmission conditions, it is coercive:

$$a(U, U) = \sum_{i \in \mathcal{M}} \int_{I_i} (\lambda_i^2 v_{ix}^2 + u_i^2 + \lambda_i^2 u_{ix}^2 + v_i^2) dx + \sum_{i, j \in \mathcal{M}} K_{ij} (u_j(N) - u_i(N))^2 .$$

Thanks to the Lax-Milgram theorem, for all $\Psi = (\psi_1, \psi_2) \in (L^2(\mathcal{A}))^2$, there exists a unique $U \in D(A_1)$ such that, for all $\bar{U} \in D(A_1)$, the following equality holds

$$a(U, \bar{U}) = \sum_{i \in \mathcal{M}} \int_{I_i} (\psi_1(\lambda_i \bar{v}_{ix} + \bar{u}_i) + \psi_2(\lambda_i \bar{u}_{ix} + \bar{v}_i)) ;$$

by using Lemma 4.1 we obtain $(I - A_1)U = \Psi$ a.e. .

We can conclude that A_1 is m-dissipative, then it is the generator of a contraction semigroup in Y , $\mathcal{T}_1(t)$.

From now on, we follow the path of the proof of Proposition 4.3.3 in [3] (working in E in place of X). We introduce the set

$$B_M = \{U \in C([0, T]; E) : \sup_{t \leq T} \|U(t)\|_E \leq M\}$$

equipped with the distance generated by the norm of $C([0, T]; E)$; we will find the solution to problem (4.3) as the unique fixed point in B_M of the function

$$\Phi(U) = \Phi_U(t) = \mathcal{T}_1(t)U_0 + \int_0^t \mathcal{T}_1(t-s)F(s, U(s)) \, ds .$$

$\Phi_U \in C([0, T]; E)$; moreover, thanks to the Lipschitz continuity of F in E , for $U \in B_M$, we have

$$\|\Phi_U(t)\|_E \leq \|U_0\|_E + TL_F(K)M \leq M$$

and, for $V \in B_M$,

$$\|\Phi_U(t) - \Phi_V(t)\|_E \leq L_F(K) \int_0^t \|U(s) - V(s)\|_E \, ds \leq \frac{1}{2} \sup_{[0, T]} \|U(t) - V(t)\|_E .$$

Then we are able to conclude that Φ is a contraction in B_M and it has a unique fixed point $U \in B_M$

$$U(t) = \mathcal{T}_1(t)U_0 + \int_0^t \mathcal{T}_1(t-s)F(s, U(s)) \, ds .$$

Using the above expression we deduce, for $t \in [0, T-h]$, $h > 0$,
(4.16)

$$\begin{aligned} \|U(t+h) - U(t)\|_Y &\leq \|\mathcal{T}_1(h)U_0 - U_0\|_Y + \int_0^h \|F(s, U(s))\|_Y \, ds \\ &\quad + \int_0^t ((\|f(s)\|_{H^1} + \beta)\|U(s+h) - U(s)\|_Y + \|U(s)\|_E \|f(s+h) - f(s)\|_2) \, ds, \end{aligned}$$

where $\beta = \sum_{i \in \mathcal{M}} \beta_i$. Since $f \in C([0, T]; H^1(\mathcal{A})) \cap H^1((0, T); L^2(\mathcal{A}))$, using Gronwall's lemma, we obtain

$$\|U(t+h) - U(t)\|_Y \leq C(M, K, T)h .$$

Now, using the above inequality and again the assumptions on f , we prove that

$$\|F(s+h, U(s+h)) - F(s, U(s))\|_2 \leq \overline{C}(K, M, T)h .$$

Now we can conclude as in Proposition 4.3.9 in [3], proving that U is the solution to problem (4.3), since $U_0 \in D(A_1)$. □

Remark 4.1. *It is readily seen that the solutions found in the previous two propositions verify*

$$\sup_{[0, T]} \|u_t(t)\|_2, \sup_{[0, T]} \|v_t(t)\|_2, \sup_{[0, T]} \|\phi_t(t)\|_2 \leq Q(K, M) ,$$

where Q is a quantity depending only on $a, b, \lambda_i, \beta_i, D_i$ besides to M and K .

Theorem 4.1. (Local existence) *There exists a unique local solution (u, v, ϕ) to problem (2.1)-(2.7),*

$$\begin{aligned} (u, v) &\in (C([0, T]; H^1(\mathcal{A})) \cap C^1([0, T], L^2(\mathcal{A})))^2, \\ \phi &\in C([0, T]; H^2(\mathcal{A})) \cap C^1([0, T], L^2(\mathcal{A})) . \end{aligned} \tag{4.17}$$

Moreover, $\phi \in H^1((0, T); H^1(\mathcal{A}))$.

Proof. Let $M > 2(\|u_0\|_{H^1} + \|v_0\|_{H^1})$, $K > \|\phi_0\|_{H^2} + 4M$ and $T \leq \min\{(2L_F(K))^{-1}, 1\}$. We recall that $E := \cup_{i \in \mathcal{M}} (H^1(I_i))^2$; let

$$(4.18) \quad B_{MK} = \left\{ \begin{array}{l} (u, v, \phi) \in (C([0, T]; H^1(\mathcal{A})))^2 \times C([0, T]; H^2(\mathcal{A})) : \\ \sup_{[0, T]} \|(u(t), v(t))\|_E \leq M, \sup_{[0, T]} \|\phi(t)\|_{H^2} \leq K, \\ u, \phi \in C^1([0, T]; L^2(\mathcal{A})), \sup_{[0, T]} \|u_t(t)\|_2, \sup_{[0, T]} \|\phi_t(t)\|_2 \leq Q(K, M) \end{array} \right\}.$$

We consider the function G defined in B_{MK} :

$$(u^0, v^0, \phi^0) \in B_{MK}, \quad G(u^0, v^0, \phi^0) = (u^1, v^1, \phi^1),$$

where $U^1 = (u^1, v^1)$ is the solution to (4.3) with $f = \phi_x^0$ and ϕ^1 is the solution to problem (4.4), with $g = a u^1$. The previous two propositions ensure that G is well defined from B_{MK} in B_{MK} . Let

$$\begin{aligned} (\hat{u}^0, \hat{v}^0, \hat{\phi}^0), (\bar{u}^0, \bar{v}^0, \bar{\phi}^0) &\in B_{MK}, \\ (\bar{u}^1, \bar{v}^1, \bar{\phi}^1) &= G(\bar{u}^0, \bar{v}^0, \bar{\phi}^0), (\hat{u}^1, \hat{v}^1, \hat{\phi}^1) = G(\hat{u}^0, \hat{v}^0, \hat{\phi}^0), \\ \bar{F} &= (0, \bar{\phi}_x^0 \bar{u}^1 - \beta \bar{v}^1), \hat{F} = (0, \hat{\phi}_x^0 \hat{u}^1 - \beta \hat{v}^1); \end{aligned}$$

moreover we denote by $C(M, K)$ constants depending only on the quantities K, M and on the parameters of the problem, and by $\gamma(t)$ functions of t which go to zero when t goes to zero. Then we have

$$(4.19) \quad \begin{aligned} \|\bar{U}^1(t) - \hat{U}^1(t)\|_E &= \sup_{[0, T]} \left\| \int_0^t \mathcal{T}_1(t-s) (\bar{F}(s) - \hat{F}(s)) ds \right\|_E \\ &\leq C(K, M) \int_0^T \left(\|\bar{U}^1(t) - \hat{U}^1(t)\|_E + \|\bar{\phi}^0(t) - \hat{\phi}^0(t)\|_{H^2} \right) dt, \end{aligned}$$

whence

$$(4.20) \quad \sup_{[0, T]} \|\bar{U}^1(t) - \hat{U}^1(t)\|_E \leq \gamma(T) C(M, K) \sup_{[0, T]} \|\bar{\phi}^0(t) - \hat{\phi}^0(t)\|_{H^2};$$

also, using the equations and the above inequality,

$$(4.21) \quad \sup_{[0, T]} \|\bar{u}_t^1(t) - \hat{u}_t^1(t)\|_2 \leq C(M, K) \gamma(T) \sup_{[0, T]} \|\bar{\phi}^0(t) - \hat{\phi}^0(t)\|_{H^2}.$$

Moreover, using (4.10) and the previous inequalities, we obtain

$$(4.22) \quad \sup_{[0, T]} \|\bar{\phi}^1(t) - \hat{\phi}^1(t)\|_{H^2} \leq \gamma(T) C(K, M) \sup_{[0, T]} \|\bar{\phi}^0(t) - \hat{\phi}^0(t)\|_{H^2}$$

and finally, using the equation and, again, the previous inequalities,

$$\sup_{[0, T]} \|\bar{\phi}_t^1(t) - \hat{\phi}_t^1(t)\|_2 \leq \gamma(T) C(M, K) \sup_{[0, T]} \|\bar{\phi}^0(t) - \hat{\phi}^0(t)\|_{H^2}.$$

If T is sufficiently small, then G is a contraction function in B_{MK} and let $(U, \phi) = (u, v, \phi)$ be its unique fixed point :

$$U(t) = \mathcal{T}_1(t)U_0 + \int_0^t \mathcal{T}_1(t-s)F(s, U(s)) ds,$$

$$\phi(t) = \mathcal{T}_2(t)\phi_0 + \int_0^t \mathcal{T}_2(t-s)u(s) ds .$$

Since $u \in C^1([0, T]; L^2(\mathcal{A}))$, arguing as at the end of the proof of Proposition 4.1, we prove that $\phi \in H^1((0, T); H^1(\mathcal{A}))$; thanks to the regularity properties of ϕ we can argue as at the end of the proof of Proposition 4.2 to prove that $v \in C^1([0, T]; L^2(\mathcal{A}))$. Therefore $(U, \phi) = (u, v, \phi)$ is the claimed solution. \square

5. Global existence

In this section we prove that, if the initial data are small in a suitable norm, then the local solution to problem (2.1)-(2.7) given by Theorem 4.1,

$$(5.1) \quad \begin{aligned} u, v &\in (C([0, T]; H^1(\mathcal{A})) \cap C^1([0, T]; L^2(\mathcal{A})))^2 , \\ \phi &\in C([0, T]; H^2(\mathcal{A})) \cap C^1([0, T]; L^2(\mathcal{A})) \cap H^1((0, T); H^1(\mathcal{A})) , \end{aligned}$$

can be extended to the time interval $[0, +\infty)$.

We set

$$\|f_i(t)\|_2 := \|f_i(\cdot, t)\|_{L^2(I_i)}, \quad \|f_i(t)\|_{H^s} := \|f_i(\cdot, t)\|_{H^s(I_i)}.$$

We introduce the functional

$$(5.2) \quad \begin{aligned} F_T^2(u, v, \phi) &:= \sum_{i \in \mathcal{M}} \left(\sup_{t \in [0, T]} \|u_i(t)\|_{H^1}^2 + \sup_{t \in [0, T]} \|v_i(t)\|_{H^1}^2 + \sup_{t \in [0, T]} \|\phi_i(t)\|_{H^2}^2 \right) \\ &+ \int_0^T (\|u_x(t)\|_2^2 + \|v(t)\|_{H^1}^2 + \|v_t(t)\|_2^2 + \|\phi_x(t)\|_{H^1}^2 + \|\phi_{xt}(t)\|_2^2) dt ; \end{aligned}$$

we are able to prove that the functional satisfies the following inequality

$$(5.3) \quad F_T^2(u, v, \phi) \leq c_1 F_0^2(u, v, \phi) + c_2 F_T^3(u, v, \phi) , \quad c_1, c_2 > 0 ,$$

then, by standard arguments [20], if $F_0(u, v, \phi)$ is small then $F_T(u, v, \phi)$ remains bounded for all $T > 0$.

The above inequality turns out to be true by some a priori estimates holding for a local solution (5.1), as we are going to prove in the following propositions. As in the previous section, in order to simplify the notations, all the proofs are given in the case of a graph composed by a single node N and m arcs I_i connecting that node to the external points a_i , $i \in \mathcal{M} = \{1, 2, \dots, m\}$.

Proposition 5.1. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$(5.4) \quad \begin{aligned} &\sum_{i \in \mathcal{M}} \left(\sup_{[0, T]} \|u_i(t)\|_2^2 + \sup_{[0, T]} \|v_i(t)\|_2^2 + \beta_i \int_0^T \|v_i(t)\|_2^2 dt \right) \\ &\leq C \sum_{i \in \mathcal{M}} \left(\|u_{0i}\|_2^2 + \|v_{0i}\|_2^2 + \sup_{[0, T]} \|u_i(t)\|_{H^1} \int_0^T (\|\phi_{ix}(t)\|_2^2 + \|v_i(t)\|_2^2) dt \right) \end{aligned}$$

for a suitable positive constant C .

Proof. We multiply the first equation in (2.1) by u_i , the second one by v_i and we sum them; after summing up for $i \in \mathcal{M}$, for $\tau \leq T$ we obtain

$$\begin{aligned}
 (5.5) \quad & \sum_{i \in \mathcal{M}} \left(\|u_i(\tau)\|_2^2 + \|v_i(\tau)\|_2^2 + \beta_i \int_0^\tau \|v_i(t)\|_2^2 dt \right) \\
 & \leq -C_1 \int_0^\tau \left(\sum_{i \in \mathcal{I}} \lambda_i u_i(N, t) v_i(N, t) - \sum_{i \in \mathcal{O}} \lambda_i u_i(N, t) v_i(N, t) \right) dt \\
 & + C_2 \sum_{i \in \mathcal{M}} \left(\|u_{0i}\|_2^2 + \|v_{0i}\|_2^2 + \sup_{[0, T]} \|u_i(t)\|_{H^1} \int_0^T (\|\phi_{ix}(t)\|_2^2 + \|v_i(t)\|_2^2) dt \right)
 \end{aligned}$$

for suitable positive constants C_1, C_2 . The transmission conditions (2.6) imply that the term at node is non positive (see Section 3), then the claim. \square

Proposition 5.2. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$\begin{aligned}
 (5.6) \quad & \sum_{i \in \mathcal{M}} \left(\sup_{[0, T]} \|v_{ix}(t)\|_2^2 + \sup_{[0, T]} \|v_{it}(t)\|_2^2 + \int_0^T \|v_{it}(t)\|_2^2 dt \right) \\
 & \leq C (\|v_0\|_{H^1}^2 + \|u_0\|_{H^1}^2 \|\phi_0\|_{H^2}^2) \\
 & + C \sum_{i \in \mathcal{M}} \sup_{[0, T]} \|u_i(t)\|_{H^1} \int_0^T (\|\phi_{ixt}(t)\|_2^2 + \|v_{it}(t)\|_2^2) dt \\
 & + C \sum_{i \in \mathcal{M}} \sup_{[0, T]} \|\phi_x(t)\|_{H^1} \int_0^T (\|v_{it}(t)\|_2^2 + \|v_i(t)\|_{H^1}^2) dt
 \end{aligned}$$

for a suitable positive constant C .

Proof. In order to avoid the presence of the traces at the node N of the functions $u_{it}, v_{it}, \phi_{it}$ in the following calculations, we must not derive in time the equations, so we introduce the difference $\Delta^h f(x, t) = f(x, t+h) - f(x, t)$; we have, for $i \in \mathcal{M}$,

$$(5.7) \quad \begin{cases} (\Delta^h u_{it} + \lambda_i \Delta^h v_{ix}) \Delta^h u_i = 0, \\ (\Delta^h v_{it} + \lambda_i \Delta^h u_{ix}) \Delta^h v_i = (\Delta^h (u_i \phi_{ix}) - \beta_i \Delta^h v_i) \Delta^h v_i. \end{cases}$$

Summing the above two equations and integrating over $I_i \times (\delta, \tau)$, for $0 < \delta < \tau < T$, $|h| \leq \min\{\delta, T - \tau\}$ we obtain

$$\begin{aligned}
 (5.8) \quad & \int_\delta^\tau \int_{I_i} \partial_t \left(\frac{(\Delta^h u_i)^2 + (\Delta^h v_i)^2}{2} \right) dx dt + \int_\delta^\tau \int_{I_i} \lambda_i \partial_x (\Delta^h v_i \Delta^h u_i) dx dt \\
 & = \int_\delta^\tau \int_{I_i} (\Delta^h (u_i \phi_{ix}) \Delta^h v_i - \beta_i (\Delta^h v_i)^2) dx dt.
 \end{aligned}$$

Using the boundary conditions (2.3) and the transmission conditions (2.6) we can compute

$$\sum_{i \in \mathcal{M}} \int_\delta^\tau \int_{I_i} \lambda_i \partial_x (\Delta^h v_i \Delta^h u_i) dx dt = \frac{1}{2} \int_\delta^\tau \sum_{i, j \in \mathcal{M}} K_{ij} (\Delta^h u^j(N, t) - \Delta^h u_i(N, t))^2 dt \geq 0.$$

Now we divide the equalities (5.8) by h^2 , we sum them for $i \in \mathcal{M}$ and, letting first h and then δ go to zero, we obtain

$$\begin{aligned}
 & \sum_{i \in \mathcal{M}} \left(\|v_{ix}(\tau)\|_2^2 + \|v_{it}(\tau)\|_2^2 + \beta_i \int_0^\tau \|v_{it}(t)\|_2^2 \right) dt \\
 & \leq C_1 \sum_{i \in \mathcal{M}} \|v_{0ix}\|_2^2 + \|v_{it}(0)\|_2^2 \\
 (5.9) \quad & + C_2 \sum_{i \in \mathcal{M}} \sup_{[0,T]} \|u_i(t)\|_{H^1} \int_0^T (\|\phi_{ixt}(t)\|_2^2 + \|v_{it}(t)\|_2^2) dt \\
 & + C_3 \sup_{[0,T]} \|\phi_x(t)\|_{H^1} \int_0^T (\|v_{it}(t)\|_2^2 + \|v_i(t)\|_{H^1}^2) dt
 \end{aligned}$$

for suitable positive constant C_i . □

Proposition 5.3. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$\begin{aligned}
 (5.10) \quad & \sum_{i \in \mathcal{M}} \sup_{[0,T]} \|u_{ix}(t)\|_2^2 \leq C \sum_{i \in \mathcal{M}} \left(\sup_{[0,T]} \|v_{it}(t)\|_2^2 + \sup_{[0,T]} \|v_i(t)\|_2^2 \right) \\
 & + C \sum_{i \in \mathcal{M}} \sup_{[0,T]} \|u_i(t)\|_{H^1} \left(\sup_{[0,T]} \|u_{ix}(t)\|_2^2 + \sup_{[0,T]} \|\phi_{ix}(t)\|_2^2 \right)
 \end{aligned}$$

for a suitable positive constant C .

Proof. We multiply the second equation by u_{ix} , we integrate over I_i and we sum for $i \in \mathcal{M}$; using the Cauchy-Schwartz inequality, we obtain the claim □

Proposition 5.4. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$\begin{aligned}
 (5.11) \quad & \sum_{i \in \mathcal{M}} \int_0^T \|u_{ix}(t)\|_2^2 dt \leq C \sum_{i \in \mathcal{M}} \int_0^T (\|v_{it}(t)\|_2^2 + \|v_i(t)\|_2^2) dt \\
 & + C \sum_{i \in \mathcal{M}} \sup_{[0,T]} \|u_i(t)\|_{H^1} \int_0^T (\|u_{ix}(t)\|_2^2 + \|\phi_{ix}(t)\|_2^2) dt
 \end{aligned}$$

for a suitable positive constant C .

Proof. We multiply the second equation by u_{ix} , we integrate over $I_i \times (0, T)$ and we sum for $i \in \mathcal{M}$; using the Cauchy-Schwartz inequality, we obtain the claim. □

Proposition 5.5. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$\begin{aligned}
(5.12) \quad & \sum_{i \in \mathcal{M}} \int_0^T \|v_{ix}(t)\|_2^2 dt \leq C \sum_{i \in \mathcal{M}} (\|v_{0i}\|_2^2 + \|u_{0i}\|_{H^1}^2 (1 + \|\phi_{0i}\|_{H^1}^2)) \\
& + C \sum_{i \in \mathcal{M}} \left(\sup_{[0,T]} \|v_{it}(t)\|_2^2 + \int_0^T \|v_{it}(t)\|_2^2 dt \right) \\
& + C \sum_{i \in \mathcal{M}} \left(\sup_{[0,T]} \|u_i(t)\|_{H^1} + \sup_{[0,T]} \|\phi_{ix}(t)\|_{H^1} \right) \int_0^T (\|v_i(t)\|_{H^1}^2 + \|\phi_{ix}(t)\|_2^2) dt
\end{aligned}$$

for a suitable positive constant C .

Proof. Using the same notations of the proof of Proposition 5.2, by the second equation in (2.1) we obtain, for $0 < \delta < \tau < T$, $|h| \leq \min\{\delta, T - \tau\}$,

$$\begin{aligned}
(5.13) \quad & \int_\delta^\tau \int_{I_i} ((v_i \Delta^h v_i)_t - v_{it} \Delta^h v_i - \lambda_i v_{ix} \Delta^h u_i) dx dt \\
& + \int_\delta^\tau \int_{I_i} \lambda_i (v_i \Delta^h u_i)_x = \int_\delta^\tau \int_{I_i} v_i (\Delta^h (u_i \phi_{ix}) - \beta_i \Delta^h v_i) dx dt .
\end{aligned}$$

Using conditions (2.3) and (2.6) we can write

$$\begin{aligned}
(5.14) \quad & \lim_{h \rightarrow 0} \sum_{i \in \mathcal{M}} \frac{1}{h} \int_\delta^\tau \int_{I_i} (-\lambda_i v_{ix} \Delta^h u_i + \beta_i v_i \Delta^h v_i) dx dt \\
& = \lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \in \mathcal{M}} \int_{I_i} (-v_i(x, t) \Delta^h v_i(x, t) dx + v_i(x, 0) \Delta^h v_i(x, 0)) dx \\
& + \lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \in \mathcal{M}} \int_\delta^\tau \int_{I_i} (v_{it} \Delta^h v_i + v_i (\Delta^h (u_i \phi_{ix}))) dx dt \\
& - \lim_{h \rightarrow 0} \frac{1}{h} \sum_{i, j \in \mathcal{M}} \frac{K_{ij}}{2} (u_j(N, t) - u_i(N, t)) \Delta^h (u_j(N, t) - u_i(N, t)) dt .
\end{aligned}$$

In order to treat the last term we set

$$H(t) = u_j(N, t) - u_i(N, t)$$

and using the continuity of the above function we have

$$\begin{aligned}
(5.15) \quad & \lim_{h \rightarrow 0} \frac{1}{h} \int_\delta^\tau H(t) \Delta^h H(t) dt = \lim_{h \rightarrow 0} \frac{1}{2h} \int_\delta^\tau H(t) (\Delta^h H(t) - \Delta^{-h} H(t)) dt \\
& = \lim_{h \rightarrow 0} \frac{1}{2h} \left(- \int_{\delta-h}^\delta H(t) H(t+h) dt + \int_{\tau-h}^\tau H(t) H(t+h) dt \right) \\
& = \frac{1}{2} (H^2(\tau) - H^2(\delta)) ,
\end{aligned}$$

Now we obtain the claim letting h and then δ go to zero in (5.13), using equality (5.15). \square

Proposition 5.6. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$(5.16) \quad \begin{aligned} & \sum_{i \in \mathcal{M}} \left(\sup_{[0, T]} \|\phi_{it}(t)\|_2^2 + \int_0^T (\|\phi_{it}(t)\|_2^2 + \|\phi_{itx}(t)\|_2^2) dt \right) \\ & \leq C \sum_{i \in \mathcal{M}} \left(\|\phi_{0i}\|_{H^2}^2 + \|u_{0i}\|_2^2 + \int_0^T \|v_{ix}(t)\|_2^2 dt \right), \end{aligned}$$

for a suitable positive constant C .

Proof. Using the same notations of the proof of Proposition 5.2, by the third equation in (2.1) we obtain, for all $i \in \mathcal{M}$,

$$(5.17) \quad \begin{aligned} & \sum_{i \in \mathcal{M}} \int_{I_i} \int_{\delta}^{\tau} \frac{((\Delta^h \phi_i)^2)_t}{2} dx dt = \\ & \sum_{i \in \mathcal{M}} D_i \int_{\delta}^{\tau} \int_{I_i} ((\Delta^h \phi_{ix})(\Delta^h \phi_i))_x dx dt - \sum_{i \in \mathcal{M}} D_i \int_{\delta}^{\tau} \int_{I_i} (\Delta^h \phi_{ix})^2 dx dt \\ & + \sum_{i \in \mathcal{M}} \int_{\delta}^{\tau} \int_{I_i} (a \Delta^h u_i)(\Delta^h \phi_i) - b(\Delta^h \phi_i)^2 dx dt. \end{aligned}$$

We denote by B^N the first term on the right hand side; using the transmission conditions (2.5) we have

$$(5.18) \quad \begin{aligned} B^N &= \int_{\delta}^{\tau} \left(\sum_{i \in \mathcal{I}} D_i \Delta^h \phi_i \Delta^h \phi_{ix}(N, t) - \sum_{i \in \mathcal{O}} D_i \Delta^h \phi_i \Delta^h \phi_{ix}(N, t) \right) dt \\ &= -\frac{1}{2} \sum_{j, i \in \mathcal{M}} \int_{\delta}^{\tau} \alpha_{ij} (\Delta^h \phi_j(N, t) - \Delta^h \phi_i(N, t))^2 dt. \end{aligned}$$

We divide equation (5.17) by h^2 and using the Cauchy-Schwartz inequality and letting, first h and then δ , go to zero we obtain

$$(5.19) \quad \begin{aligned} & \sum_{i \in \mathcal{M}} \|\phi_{it}(\tau)\|_2^2 + \sum_{i \in \mathcal{M}} \int_0^{\tau} (\|\phi_{it}(t)\|_2^2 + \|\phi_{itx}(t)\|_2^2) dt \\ & \leq C \sum_{i \in \mathcal{M}} \left(\|\phi_{it}(0)\|_2^2 + \int_0^T \|u_{it}(t)\|_2^2 dt \right) dt. \end{aligned}$$

□

Proposition 5.7. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); then*

$$(5.20) \quad \sum_{i \in \mathcal{M}} \left(\sup_{[0, T]} \|\phi_{ixx}(t)\|_2^2 + \sup_{[0, T]} \|\phi_{ix}(t)\|_2^2 \right) \leq C \sum_{i \in \mathcal{M}} \left(\sup_{[0, T]} \|\phi_{it}(t)\|_2^2 + \sup_{[0, T]} \|u_i(t)\|_2^2 \right).$$

Proof. We multiply the third equation in (2.1) by $D_i \phi_{ixx}$, then we sum over $i \in \mathcal{M}$ and using the Cauchy-Schwartz inequality and the boundary conditions we obtain

$$(5.21) \quad \begin{aligned} & \sum_{i \in \mathcal{M}} (\|\phi_{ixx}(\tau)\|_2^2 + \|\phi_{ix}(\tau)\|_2^2) \leq C \sum_{i \in \mathcal{M}} (\|\phi_{it}(\tau)\|_2^2 + \|u_i(\tau)\|_2^2) \\ & + C \left(\sum_{i \in \mathcal{I}} D_i \phi_i \phi_{ix}(N, \tau) - \sum_{i \in \mathcal{O}} D_i \phi_i \phi_{ix}(N, \tau) \right) dt; \end{aligned}$$

again, using the transmission conditions (2.5), we are able to show that the last term is non positive, so we have the claim. \square

Proposition 5.8. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7); moreover, for all $\nu \in \mathcal{P}$ let $K_{kj}^\nu \neq 0$ in (2.6) for, at least, one $k \in \mathcal{M}^\nu$ and for all $j \in \mathcal{M}^\nu$. Then*

$$(5.22) \quad \begin{aligned} & \sum_{i \in \mathcal{M}} \int_0^T (\|\phi_{ix}(t)\|_2^2 + \|\phi_{ixx}(t)\|_2^2) dt \\ & \leq C \sum_{i \in \mathcal{M}} \int_0^T (\|u_{ix}(t)\|_2^2 + \|v_i(t)\|_{H^1} + \|\phi_{it}(t)\|_2) dt . \end{aligned}$$

This proposition is crucial in treating the nonlinear terms $\phi_{ix}u_i$, since it provides an L^∞ -norm's estimate for ϕ_{ix} ; in the proof, which is given (as for the other propositions) in the case of a single node, we will be leaded to consider the transmission term

$$\int_0^T \sum_{i \in \mathcal{I}} \left(D_i u_i(N, t) \phi_{ix}(N, t) - \sum_{i \in \mathcal{O}} D_i u_i(N, t) \phi_{ix}(N, t) \right) dt$$

which cannot be discarded by means of sign properties, so we are going to use conditions at node to obtain a suitable estimate for it. In the case of two arcs, conditions (2.6) imply that the quantity $u_2(N, t) - u_1(N, t)$ is proportional to $v_1(N, t)$ (and to $v_2(N, t)$); then, thanks to conditions (2.6), we have

$$(5.23) \quad \begin{aligned} & \left| \int_0^T (D_1 u_1 \phi_{1x}(N, t) - D_2 u_2 \phi_{2x}(N, t)) dt \right| \\ & \leq C \int_0^T \|v_1(t)\|_\infty \|\phi_{1x}\|_\infty dt \leq C \int_0^T \left(\frac{\|v_1(t)\|_{H^1}^2}{2\epsilon} + \frac{\epsilon \|\phi_{1x}(T)\|_{H^1}^2}{2} \right) dt . \end{aligned}$$

The further conditions on coefficients K_{jk} assumed in the claim of the Proposition are necessary to extend this estimate to the cases when the number of arcs is greater than two. We need the following lemma.

Lemma 5.1. *Let (u, v, ϕ) be a local solution (5.1) to problem (2.1)-(2.7) in the case of a single node N ; moreover, let $K_{kj} \neq 0$ in (2.6) for, at least, one $k \in \mathcal{M}$ and for all $j \in \mathcal{M}$. Then, for suitable θ_i^j ,*

$$u_j(N, t) = u_k(N, t) + \sum_{i \neq k} \theta_i^j v_i(N, t) \quad \text{for all } j \in \mathcal{M}.$$

Proof. Let $\gamma_i = 1$ for $i \in \mathcal{O}$ and $\gamma_i = -1$ otherwise; for $i \neq k$ we consider the $m - 1$ transmission relations

$$(5.24) \quad \begin{aligned} \gamma_i \lambda_i v_i(N) &= \sum_{j \in \mathcal{M}, j \neq i} K_{ij} (u_j(N) - u_i(N)) \\ &= \sum_{j \in \mathcal{M}, j \neq i, k} K_{ij} (u_j(N) - u_k(N)) - \left(\sum_{j \in \mathcal{M}, j \neq i} K_{ij} \right) (u_i(N) - u_k(N)) \end{aligned}$$

which constitute a linear system in the unknowns $(u_j - u_k)$, $j \neq k$. The assumptions on K_{kj} ensure that the matrix of the coefficients is non singular (if $k = 1$ it is

immediate to check that it has strictly dominant diagonal); then the claim follows. \square

Let notice that it is never possible to write $u_i(N, t)$ for all $i \in \mathcal{M}$ as linear combination of $v_i(N, t)$, since the matrix of the linear system (2.6) is singular .

Proof. (Proposition 5.8)

We multiply the third equation in (2.1) by $D_i \phi_{i_{xx}}$; after summing for $i \in \mathcal{M}$, using the Cauchy-Schwartz inequality and the boundary conditions, we obtain

$$\begin{aligned}
 (5.25) \quad & \sum_{i \in \mathcal{M}} \int_0^T \left(\|\phi_{i_x}(t)\|_2^2 + \|\phi_{i_{xx}}(t)\|_2^2 \right) dt \leq C \sum_{i \in \mathcal{M}} \int_0^T \left(\|u_{i_x}(t)\|_2^2 + \|\phi_{i_t}(t)\|_2^2 \right) dt \\
 & + C_1 \sum_{i \in \mathcal{M}} D_i \int_0^\tau \int_{I_i} ((-a u_i + b \phi_i) \phi_{i_x})_x dx dt \\
 & = C \sum_{i \in \mathcal{M}} \int_0^T \left(\|u_{i_x}(t)\|_2^2 + \|\phi_{i_t}(t)\|_2^2 \right) dt + C_1 \sum_{i, j \in \mathcal{M}} \alpha_{ij} \int_0^\tau -b (\phi_j - \phi_i)^2(N, t) dt \\
 & - C_1 a \int_0^\tau \left(\sum_{i \in \mathcal{I}} D_i u_i(N, t) \phi_{i_x}(N, t) - \sum_{i \in \mathcal{O}} D_i u_i(N, t) \phi_{i_x}(N, t) \right) dt ,
 \end{aligned}$$

where we used the transmission conditions (2.5). Thanks to the last lemma we have

$$\begin{aligned}
 (5.26) \quad & \sum_{i \in \mathcal{I}} D_i u_i(N, t) \phi_{i_x}(N, t) - \sum_{i \in \mathcal{O}} D_i u_i(N, t) \phi_{i_x}(N, t) \\
 & = \sum_{i \in \mathcal{I}} \left(u_k(N) + \sum_{j \neq k} \theta_j^i v_j(N) \right) D_i \phi_{i_x}(N, t) \\
 & - \sum_{i \in \mathcal{O}} \left(u_k(N) + \sum_{j \neq k} \theta_j^i v_j(N) \right) D_i \phi_{i_x}(N, t) ;
 \end{aligned}$$

now we use condition (3.1) to discard the terms containing $u_k(N)$ and we obtain an estimate similar to (5.23). Then the claim follows. \square

Now we notice that arranging the results obtained in this section, the quadratic terms (not involving the initial data) in the estimates of Propositions 5.1 - 5.8 can be bounded by means of cubic ones; so, collecting all the estimates, we obtain the inequality (5.3) holding for the functional F introduced at the beginning of the section, for all $T > 0$. Then we can conclude, as in [20, 12], that, for F_0 suitable small , F remains bounded for all $T > 0$; this fact proves the following theorem.

Theorem 5.1. (Global existence) *Let $K_{kj} \neq 0$ in (2.6) for (at least) one $k \in \mathcal{M}$ and for all $j \in \mathcal{M}$, $j \neq k$. There exists $\epsilon_0 > 0$ such that, for*

$$\|u_0\|_{H^1}, \|v_0\|_{H^1}, \|\phi_0\|_{H^2} \leq \epsilon_0 ,$$

there exists a unique global solution (u, v, ϕ) to problem (2.1)-(2.7),

$$(u, v) \in C([0, +\infty); H^1(\mathcal{A})) \cap C^1([0, +\infty); L^2(\mathcal{A})) ,$$

$$\phi \in C([0, +\infty); H^2(\mathcal{A})) \cap C^1([0, +\infty); L^2(\mathcal{A})) \cap H^1((0, +\infty); H^1(\mathcal{A})) .$$

Moreover, $F_T(u, v, \phi)$ is bounded, uniformly in T .

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